

## FRACTIONAL ABSOLUTE MOMENTS OF HEAVY TAILED DISTRIBUTIONS

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ABSTRACT. Several convenient methods for calculation of fractional absolute moments are given with application to heavy tailed distributions. We use techniques of fractional differentiation to obtain formulae for  $E[|X - \mu|^\gamma]$  with  $1 < \gamma < 2$  and  $\mu \in \mathbb{R}$ . The main focus is on heavy tailed distributions, several examples are given with analytical expressions of fractional absolute moments. As applications, we calculate the fractional moment errors for both prediction and parameter estimation problems.

## 1. FRACTIONAL MOMENTS

In many diverse areas an important task is to predict future random elements based on previous available observations. For linear time series models, e.g. ARMA, ARIMA or FARIMA models, linear predictors given by linear combinations of past observations are often considered. More sophisticated statistical methods are used for prediction in non-linear models, such as GARCH or SV and their variations, which have found applications in many different areas. In general, the optimal predictor minimizes the expected loss function that measures a distance between the predicted and the actual value of the series. The most popular is the  $L^2$  loss function since it is easily tractable and intuitively clear. In that case, the best predictor, in the sense of minimizing the mean squared error, is the conditional expectation. The mean squared error indicates the uncertainty in the prediction. However, this measure is not applicable when concerning random elements with no finite second moments, which are often observed in the reality.

The same problem arises in the parameter estimation where the quality of estimators is traditionally measured by the mean squared error. The estimator directly reflected such a criterion is the ordinary least square estimator (OLS) in linear regression analysis, which minimizes the  $L^2$  vertical distance between the observed data and the assumed linear model. A large number of related estimators have been invented and compared with OLS by means of  $L^2$  distances between the true parameter and proposed estimators. However, once we lose the finite second moment condition for random components of the estimators, we could not use the mean squared error to judge the goodness of estimators. Then alternative measures are required.

Although there are many potential loss functions, the  $L^p$  loss function with  $0 < p < 2$  is a natural plausible candidate to evaluate the goodness of prediction or estimators. Thus, we have to study the fractional absolute moments  $m_p := E[|X|^p]$  of order  $0 < p < 2$ . We also consider  $\mu$ -centered moments  $m_{\mu,p} := E[|X - \mu|^p]$  with  $\mu \in \mathbb{R}$ . As far as we know, there have not been enough researches of the fractional absolute moments except for the special case of first order absolute moment  $m_1$ . The reason is that the existing methods are unfamiliar or these methods seem to require a lot of numerical work.

Taking this into consideration, we summarize existing methods for obtaining the fractional absolute moments. In particular, we focus on the methods exploiting the Laplace (LP) transform or the characteristic function (ch.f.) of the corresponding distribution. It is well-known that the moments of integer orders are related to the derivatives of ch.f. or LP transform at zero. More generally, the theory of fractional calculus can be utilized in order to obtain the non-integer real moments. There are several works giving the relation between the fractional moments and the corresponding ch.f or LP transform. We refer to von Bahr [23], Brown [2], Ramachandran [18], Kawata [8, Sec. 11.4], Wolfe [25, 26, 27, 28], Laue [10, 11], Zolotarev [29, Sec. 2.1] and Paoletta [17, Sec. 8.3]. The methods using moment generating functions have also been studied, e.g. by Cressie et al. [4] and

Cressie and Borkent [3]. From these references, we shall explain [8], [10], [26] and [3] which are convenient for our purpose.

Our main tool is the fractional derivative of order  $\gamma = k + \lambda$  with  $k \in \mathbb{N}$ ,  $0 < \lambda < 1$ , see e.g. [10, Eq. (2.1)],

$$\frac{d^\gamma}{dt^\gamma} f(t) = \frac{d^\lambda}{dt^\lambda} f^{(k)}(t) = \frac{\lambda}{\Gamma(1-\lambda)} \int_{-\infty}^t \frac{f^{(k)}(t) - f^{(k)}(u)}{(t-u)^{1+\lambda}} du, \quad t \in \mathbb{R},$$

where  $f$  is a complex-valued function,  $f^{(k)}$  is its  $k$ th derivative and  $\Gamma$  is the Gamma function. We are mostly interested in the fractional absolute moments  $m_{1+\lambda} = E[|X|^{1+\lambda}]$  with  $0 < \lambda < 1$ . For this reason, we will need the fractional derivative of order  $1 + \lambda$  at zero,

$$(1.1) \quad \left. \frac{d^{1+\lambda}}{dt^{1+\lambda}} f(t) \right|_{t=0} = \left. \frac{d^\lambda}{dt^\lambda} f'(t) \right|_{t=0} = \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty \frac{f'(0) - f'(-u)}{u^{1+\lambda}} du.$$

The construction of our paper is as follows. In the remainder of Section 1, we make a brief survey on the relation between the fractional absolute moments and fractional derivatives using references cited above. Several convenient formulae are also derived. In Section 2 we apply the mentioned methods to the infinitely divisible distributions and examine their fractional absolute moments. Heavy tailed distributions, such as stable, Pareto, geometric stable and Linnik distributions, are considered. Especially, in Section 3 we pay attention to compound Poisson distribution that is popular in applications. In the final section, several applications are presented. The fractional errors of predictions concerning stable distributions are explicitly calculated. In addition, the estimation errors in regression models are evaluated by the fractional absolute moments in heavy tailed cases.

**1.1. Fractional derivatives of Laplace transforms.** Let  $F$  be a distribution function (d.f.) of a non-negative random variable  $X$ . Its LP transform is defined as

$$\phi(t) := \int_0^\infty e^{-tx} dF(x), \quad t \geq 0.$$

We characterize the relation between moments of  $X$  and the fractional derivative of  $\phi$ . Since  $\phi$  is only defined on the positive real axis, we define its fractional derivative of order  $1 + \lambda$ ,  $0 < \lambda < 1$ , as

$$(1.2) \quad \frac{d^{1+\lambda}}{dt^{1+\lambda}} \phi(t) = \frac{d^\lambda}{dt^\lambda} \phi'(t) = \frac{\lambda}{\Gamma(1-\lambda)} \int_t^\infty \frac{\phi'(u) - \phi'(t)}{(u-t)^{1+\lambda}} du, \quad t \geq 0.$$

We start with the following result, which is a modification of Theorem 1 in [26].

**Lemma 1.1.** *Let  $0 < \lambda < 1$  and let  $\phi$  be the LP transform of the d.f.  $F(x)$  such that  $F(x) = 0$  for  $x < 0$ . Then  $m_{1+\lambda}$  exists if and only if  $\phi'(0+)$  exists and the fractional derivative  $\left[ \frac{d^{1+\lambda}}{dt^{1+\lambda}} \phi(t) \right]_{t=0+}$  exists, in which case*

$$m_{1+\lambda} = \left[ \frac{d^{1+\lambda}}{dt^{1+\lambda}} \phi(t) \right]_{t=0+} = \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty \frac{\phi'(u) - \phi'(0+)}{u^{1+\lambda}} du.$$

*Proof.* Suppose that  $m_{1+\lambda}$  exists, then  $\phi'(u)$  exists for all  $u > 0$  and is equal to  $-\int_0^\infty x e^{-xu} dF(x)$  and  $\phi'(0+)$  exists and is equal to  $-\int_0^\infty x dF(x)$ . We substitute this into the definition (1.2) of fractional derivative and use Fubini's theorem to see that

$$\begin{aligned} \left[ \frac{d^{1+\lambda}}{dt^{1+\lambda}} \phi(t) \right]_{t=0+} &= \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty \frac{\phi'(u) - \phi'(0+)}{u^{1+\lambda}} du \\ &= \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty u^{-(1+\lambda)} \int_0^\infty x(1 - e^{-xu}) dF(x) du \end{aligned}$$

$$\begin{aligned}
&= \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty x^{1+\lambda} \int_0^\infty \frac{1-e^{-xu}}{(xu)^{1+\lambda}} x \, du \, dF(x) \\
&= \int_0^\infty x^{1+\lambda} dF(x) < \infty.
\end{aligned}$$

Conversely, the existence of  $\phi'(0+)$  implies

$$-\phi'(0+) \geq \int_0^\infty x e^{-ux} dF(x) = -\phi'(u)$$

for any  $u > 0$ . Hence, the reverse argument yields  $m_{1+\lambda} < \infty$ .  $\square$

**Remark 1.2.** (1) We applied the fractional derivative of order  $\lambda$  to the first derivative  $\phi'$ , whereas in [26, Theorem 1] the  $\lambda$ th fractional derivative of  $\phi$  was differentiated. The latter approach leads to

$$m_{1+\lambda} = \frac{d}{dt} \left[ \frac{\lambda}{\Gamma(1-\lambda)} \int_t^\infty \frac{\phi(u) - \phi(t)}{(u-t)^{1+\lambda}} du \right] \Big|_{t=0+},$$

which can be also useful representation of  $m_{1+\lambda}$ . The right-hand side can be written as

$$\frac{d}{dt} \left[ \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty \frac{\phi(u+t) - \phi(t)}{u^{1+\lambda}} du \right] \Big|_{t=0+}.$$

Since Lebesgue's dominated convergence theorem enables to make differentiation under the integral sign, we recover the formula in Lemma 1.1.

(2) In Proposition 5 of Cressie and Borkent [3], they apply the fractional integral to the derivative of moment generating function.

**1.2. Fractional derivatives of characteristic functions.** We denote the ch.f. of a random variable  $X$  with d.f.  $F$  by

$$\varphi(t) := \int_{-\infty}^\infty e^{itx} dF(x), \quad t \in \mathbb{R},$$

and denote that for  $X - \mu$  with  $\mu \in \mathbb{R}$  by

$$\varphi_\mu(t) := e^{-it\mu} \varphi(t), \quad t \in \mathbb{R}.$$

As stated before, there are series of researches regarding the relation between the fractional derivative of  $\varphi$  and the fractional absolute moment. Among them we mainly work on the result by [10] referring to others [26], [8] as supplementary results. Since the original result by [10] is quite general and theoretical, we need to fill the gap between the theory and applications.

**Lemma 1.3.** Let  $0 < \lambda < 1$  and let  $\varphi$  be the ch.f. of an arbitrary d.f.  $F$ . Then  $m_{1+\lambda}$  exists if and only if

1.  $\text{Re} \int_0^\infty \frac{\varphi'(-u)}{u^{1+\lambda}} du$  exists, and
2.  $\lim_{t \rightarrow 0+} \frac{1 - \text{Re} \varphi(t)}{t^{1+\lambda}}$  exists.

In such a case

$$(1.3) \quad m_{1+\lambda} = \frac{1}{\sin \frac{\lambda\pi}{2}} \frac{\lambda}{\Gamma(1-\lambda)} \text{Re} \int_0^\infty \frac{\varphi'(-u)}{u^{1+\lambda}} du$$

and the fractional absolute moment with center  $\mu$  ( $m_{\mu,1+\lambda} = E[|X - \mu|^{1+\lambda}]$ ) is given by

$$(1.4) \quad m_{\mu,1+\lambda} = \frac{1}{\sin \frac{\lambda\pi}{2}} \frac{\lambda}{\Gamma(1-\lambda)} \left\{ \mu \text{Im} \int_0^\infty \frac{e^{i\mu u} \varphi(-u)}{u^{1+\lambda}} du + \text{Re} \int_0^\infty \frac{e^{i\mu u} \varphi'(-u)}{u^{1+\lambda}} du \right\}.$$

*Proof.* We begin by applying equation (2.6) in [10, Theorem 2.2] with  $n = 0$  and  $h = \varphi_\mu$ , which gives

$$(1.5) \quad m_{\mu, 1+\lambda} = \frac{1}{\sin \frac{\lambda\pi}{2}} \operatorname{Re} \left[ -\frac{d^{1+\lambda}}{dt^{1+\lambda}} \varphi_\mu(t) \Big|_{t=0} \right] = \frac{1}{\sin \frac{\lambda\pi}{2}} \operatorname{Re} \left[ -\frac{d^\lambda}{dt^\lambda} \varphi'_\mu(t) \Big|_{t=0} \right].$$

Since the first derivative of the ch.f.  $\varphi_\mu$  is

$$\varphi'_\mu(u) = -i\mu e^{-i\mu u} \varphi(u) + e^{-i\mu u} \varphi'(u), \quad u \in \mathbb{R},$$

the fractional derivative of order  $1 + \lambda$  at zero, by (1.1), becomes

$$\begin{aligned} -\frac{d^\lambda}{dt^\lambda} \varphi'_\mu(t) \Big|_{t=0} &= \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty \frac{\varphi'_\mu(-u) - \varphi'_\mu(0)}{u^{1+\lambda}} du \\ &= \frac{\lambda}{\Gamma(1-\lambda)} \int_0^\infty \frac{i\mu(1 - e^{i\mu u} \varphi(-u)) + e^{i\mu u} \varphi'(-u) - \varphi'(0)}{u^{1+\lambda}} du \end{aligned}$$

and hence, by noting that  $\operatorname{Re} \varphi'(0) = 0$ ,

$$\operatorname{Re} \left[ -\frac{d^\lambda}{dt^\lambda} \varphi'_\mu(t) \Big|_{t=0} \right] = \frac{\lambda}{\Gamma(1-\lambda)} \operatorname{Re} \int_0^\infty \frac{-i\mu e^{i\mu u} \varphi(-u) + e^{i\mu u} \varphi'(-u)}{u^{1+\lambda}} du.$$

Inserting this to (1.5), we obtain the desired result (1.4). We may decompose the integral into two parts as in (1.4) since the existence of the integral

$$\operatorname{Im} \int_0^\infty \frac{e^{i\mu u} \varphi(-u)}{u^{1+\lambda}} du = \int_0^\infty \frac{\cos \mu u \operatorname{Im} \varphi(-u) + \sin \mu u \operatorname{Re} \varphi(-u)}{u^{1+\lambda}} du$$

follows from  $|\operatorname{Re} \varphi(-u)| \leq 1$  and

$$|\operatorname{Im} \varphi(-u)| \leq \int_{-\infty}^\infty |\sin(-ux)| dF(x) \wedge 1 \leq \int_{-\infty}^\infty |ux| dF(x) \wedge 1.$$

Taking  $\mu = 0$ , we get (1.3). The equivalence of the conditions 1 and 2 and the existence of  $m_{1+\lambda}$  follows from part (b) of [10, Theorem 2.2].  $\square$

Another expression of the fractional absolute moment of order  $0 < \gamma < 2$  is given in Theorem 11.4.3 of [8], which is stated as the following lemma.

**Lemma 1.4.** *A necessary and sufficient condition for the existence of  $m_\gamma$ ,  $0 < \gamma < 2$ , is that*

$$(1.6) \quad \operatorname{Re} \int_0^\infty \frac{1 - \varphi(u)}{u^{1+\gamma}} du < \infty.$$

*In this case for  $\gamma = 1 + \lambda$ ,*

$$(1.7) \quad m_{1+\lambda} = \frac{\lambda(1+\lambda)}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \operatorname{Re} \int_0^\infty \frac{1 - \varphi(u)}{u^{2+\lambda}} du.$$

**Remark 1.5.** *Equation (1.7) can also be found as (2.1.9) in [29] or (8.30) in [17], in both cases with differently written constant in front of the integral and with a typo contained. We briefly see the equivalence of Lemma 1.3 and Lemma 1.4. A simple calculation yields*

$$\begin{aligned} \int_{-\infty}^\infty |x|^{1+\lambda} dF(x) &= \frac{\lambda(1+\lambda)}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \int_{-\infty}^\infty |x|^{1+\lambda} \int_0^\infty \frac{1 - \cos ux}{u^{2+\lambda}} du dF(x) \\ (1.8) \quad &= \frac{\lambda(1+\lambda)}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \int_{-\infty}^\infty \int_0^\infty \frac{1 - \cos ux}{u^{2+\lambda}} du dF(x) \\ &= \frac{\lambda(1+\lambda)}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \int_0^\infty \frac{1 - \operatorname{Re} \varphi(u)}{u^{2+\lambda}} du, \end{aligned}$$

where we use Fubini's theorem in the last step. Hence, we obtain the first part of Lemma 1.4. Moreover, due to the integration by parts, for  $x \in \mathbb{R}$  we have

$$(1.9) \quad \int_0^\infty \frac{1 - \cos ux}{u^{2+\lambda}} du = \left[ -\frac{u^{-(1+\lambda)}}{1+\lambda} (1 - \cos ux) \right]_0^\infty + \frac{1}{1+\lambda} \int_0^\infty \frac{x \sin ux}{u^{1+\lambda}} du \\ = \lim_{u \rightarrow 0+} \frac{1}{1+\lambda} \frac{1 - \cos ux}{u^{1+\lambda}} + \frac{1}{1+\lambda} \int_0^\infty \frac{x \sin ux}{u^{1+\lambda}} du.$$

If condition (1.6) holds, then by the Lebesgue dominated convergence theorem,

$$\int_{-\infty}^\infty \lim_{u \rightarrow 0+} \frac{1}{1+\lambda} \frac{1 - \cos ux}{u^{1+\lambda}} dF(x) = \frac{1}{1+\lambda} \lim_{u \rightarrow 0+} \frac{1 - \operatorname{Re} \varphi(u)}{u^{1+\lambda}} < \infty,$$

and by Fubini's theorem,

$$\frac{1}{1+\lambda} \int_{-\infty}^\infty \int_0^\infty \frac{x \sin ux}{u^{1+\lambda}} du dF(x) = \frac{1}{1+\lambda} \int_0^\infty \frac{\operatorname{Re} \varphi'(-u)}{u^{1+\lambda}} du < \infty.$$

Thus, conditions 1 and 2 in Lemma 1.3 are satisfied. We can prove the converse in a similar manner. Finally, if  $m_{1+\lambda} < \infty$  then

$$\lim_{u \rightarrow 0+} \frac{1 - \operatorname{Re} \varphi(u)}{u^{1+\lambda}} = 0$$

follows from (1.6). This, together with (1.8) and (1.9), yields the expression (1.3) in Lemma 1.3.

Moreover, Kawata [8, Theorem 11.4.4] has obtained expressions for  $m_\gamma$ ,  $2 < \gamma$ , in the form of

$$m_\gamma = C_\ell \int_0^\infty u^{-(1+\lambda)} \left[ 1 - \operatorname{Re} \varphi(t) + \sum_{k=1}^\ell \frac{u^{2k}}{(2k)!} \varphi^{(2k)}(0) \right] du,$$

where  $\ell \in \mathbb{N}$  is such that  $2\ell < \gamma < 2\ell + 2$  and  $C_\ell$  is a positive constant depending on  $\ell$ . In other context, Wolfe [26] has derived techniques to obtain moments of any rational order from the ch.f. which are in different manner. It is well known, see e.g. [15], that the absolute  $n$ th moment is obtained from the identity

$$\frac{1}{\pi} \lim_{a \rightarrow \infty} \int_{-a}^a \frac{\sin xt}{t} dt = \operatorname{sign}(x), \quad x \in \mathbb{R},$$

which yields, with applications of Fubini's theorem, that

$$\int_{-\infty}^\infty |x|^n dF(x) = \int_{-\infty}^\infty x^n \operatorname{sign}(x) dF(x) = \frac{1}{2\pi i^{n+1}} \int_{-\infty}^\infty [\varphi^{(n)}(u) + \varphi^{(n)}(-u)] \frac{du}{u}.$$

Wolfe [26] generalized this relation for calculating  $m_\gamma$ ,  $\gamma \in \mathbb{R}$ .

## 2. INFINITELY DIVISIBLE DISTRIBUTIONS

In this section, we examine the class of infinitely divisible (ID for short) distributions, whose general definitions and many distributional properties are given by their ch.f. Many well-known distributions belong to this class and there are magnitude of applications in different areas (finance, insurance, physics, astronomy etc.). Here we work on the distribution without Gaussian part, its ch.f. is

$$(2.1) \quad \varphi(t) = \exp \left\{ i\delta u + \int_{\mathbb{R}} (e^{itx} - 1 - itx \mathbf{1}_{\{x \leq 1\}}) \nu(dx) \right\}, \quad t \in \mathbb{R},$$

where  $\delta \in \mathbb{R}$  is a centering constant and  $\nu$  is the Lévy measure satisfying  $\nu(\{0\}) = 0$  and  $\int_{\mathbb{R}} (|x|^2 \wedge 1) \nu(dx) < \infty$ . For more details on the definition and properties, we refer to Sato [21].

Although we can not calculate  $m_\gamma$  from density functions, because they are not available for most ID distributions, we can directly apply the fractional derivative to ch.f. and obtain fractional

moments. An advantage is that we can check the existence of fractional moments by the Lévy measure of ID distributions and we need to check neither condition 1 nor condition 2 of Lemma 1.3. The following result is well-known criteria for moments (see e.g. [21, Corollary 25.8] or [24, Theorem 2]). In our case of interest  $E[|X|^\gamma]$ ,  $0 < \gamma < 2$ , we find a simple proof and give it in Appendix.

**Lemma 2.1.** *Let  $X$  be an ID distribution with Lévy measure  $\nu$ . Then for  $0 < \gamma < 2$ ,  $m_\gamma < \infty$  if and only if*

$$\int_{|x|>1} |x|^\gamma \nu(dx) < \infty.$$

In what follows, we present the examples.

**2.1. Stable distributions.** As a representative of heavy tailed distributions we firstly consider stable distributions. A random variable  $X$  has a stable distribution with parameters  $0 < \alpha \leq 2$ ,  $\sigma \geq 0$ ,  $-1 \leq \beta \leq 1$  and  $\delta \in \mathbb{R}$  if its ch.f. has the form, cf. [20, Definition 1.1.6],

$$(2.2) \quad \varphi(t) = \exp \{ i\delta t - \sigma^\alpha |t|^\alpha \omega(t) \}, \quad t \in \mathbb{R},$$

where

$$(2.3) \quad \omega(t) = \begin{cases} 1 - i\beta \tan \frac{\pi\alpha}{2} \text{sign}(t), & \text{if } \alpha \neq 1 \\ 1 + i\beta \frac{2}{\pi} \text{sign}(t) \log |t|, & \text{if } \alpha = 1. \end{cases}$$

It is well-known that if  $\gamma < \alpha < 2$ , the moment of order  $\gamma$  exists, otherwise it does not exist, see e.g. [18, Sec. 4] or [20, Property 1.2.16]. We briefly review the existing results on the moments. If  $0 < \alpha < 1$  and  $X$  is a stable subordinator with the LP transform given by  $E[e^{-uX}] = \exp\{-\sigma^\alpha u^\alpha\}$ , then for  $-\infty < \gamma < \alpha$ ,

$$E[X^\gamma] = \frac{\Gamma(1 - \gamma/\alpha)}{\Gamma(1 - \gamma)} \sigma^\gamma,$$

which is shown by Wolfe [26, Sec. 4] or Shanbhag and Sreehari [22]. In symmetric case ( $\beta = 0$ ) with  $\delta = 0$ , it is shown in Shanbhag and Sreehari [22, Theorem 3] that

$$(2.4) \quad m_\gamma = \frac{2^\gamma \Gamma((1 + \gamma)/2) \Gamma(1 - \gamma/\alpha)}{\Gamma(1 - \gamma/2) \Gamma(1/2)} \sigma^\gamma, \quad -1 < \gamma < \alpha,$$

where they rely on the decomposition of the symmetric stable distribution (see also Section 25 in Sato [21]). For general  $\beta$  and  $\delta = 0$ , the following relation is proved by two different methods in Section 8.3 of [17], see also [20, p. 18],

$$(2.5) \quad m_\gamma = \kappa^{-1} \Gamma(1 - \frac{\gamma}{\alpha}) (1 + \theta^2)^{\frac{\gamma}{2\alpha}} \cos\left(\frac{\gamma}{\alpha} \arctan \theta\right) \sigma^\gamma, \quad -1 < \gamma < \alpha,$$

where  $\theta = \beta \tan \frac{\pi\alpha}{2}$  and

$$\kappa = \begin{cases} \Gamma(1 - \gamma) \cos \frac{\gamma\pi}{2}, & \text{if } \gamma \neq 1, \\ \frac{\pi}{2}, & \text{if } \gamma = 1. \end{cases}$$

Using the fractional derivative, we obtain from Lemma 1.3 not only another proof of (2.5) but also formulae for fractional absolute  $\mu$ -centered moments which seem to be new.

**Proposition 2.2.** *Let  $X$  have a stable distribution with real parameters  $\alpha > 1$ ,  $|\beta| \leq 1$ ,  $\delta = 0$  and  $\sigma > 0$ . Then, for  $0 < \lambda < \alpha - 1$ , we have*

$$(2.6) \quad m_{1+\lambda} = \frac{\lambda \Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2}) \Gamma(1 - \lambda)} \sigma^{1+\lambda} (1 + \theta^2)^{\frac{1+\lambda}{2\alpha} - \frac{1}{2}} \\ \times \left[ \cos \left\{ \left(1 - \frac{1+\lambda}{\alpha}\right) \arctan \theta \right\} + \theta \sin \left\{ \left(1 - \frac{1+\lambda}{\alpha}\right) \arctan \theta \right\} \right],$$

and for  $\mu \in \mathbb{R}$ ,

$$(2.7) \quad m_{\mu,1+\lambda} = \frac{\lambda}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left\{ \mu \int_0^\infty u^{-(1+\lambda)} e^{-\sigma^\alpha u^\alpha} \sin(\mu u - \theta \sigma^\alpha u^\alpha) du \right. \\ \left. + \alpha \sigma^\alpha \int_0^\infty u^{\alpha-\lambda-2} e^{-\sigma^\alpha u^\alpha} \left[ \cos(\mu u - \theta \sigma^\alpha u^\alpha) - \theta \sin(\mu u - \theta \sigma^\alpha u^\alpha) \right] du \right\},$$

where  $\theta = \beta \tan \frac{\pi\alpha}{2}$ . If  $X$  is symmetric ( $\beta = 0$ ), it follows that

$$(2.8) \quad m_{1+\lambda} = \frac{\lambda \Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \sigma^{1+\lambda}$$

and

$$(2.9) \quad m_{\mu,1+\lambda} = \frac{\lambda \sigma^{1+\lambda}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left\{ \frac{\mu}{\sigma} \int_0^\infty u^{-(1+\lambda)} e^{-u^\alpha} \sin\left(\frac{\mu u}{\sigma}\right) du + \alpha \int_0^\infty u^{\alpha-\lambda-2} e^{-u^\alpha} \cos\left(\frac{\mu u}{\sigma}\right) du \right\}.$$

*Proof.* We begin with the expression of  $m_{\mu,1+\lambda}$ . Let  $\varphi(u)$  be the ch.f. of a stable distribution with  $\delta = 0$  and  $\alpha > 1$ . Since we have, for  $u > 0$ ,

$$\begin{aligned} \operatorname{Im} e^{i\mu u} \varphi(-u) &= \exp(-\sigma^\alpha u^\alpha) \sin(\mu u - \theta \sigma^\alpha u^\alpha), \\ \operatorname{Re} e^{i\mu u} \varphi'(-u) &= \alpha \sigma^\alpha u^{\alpha-1} \exp(-\sigma^\alpha u^\alpha) \cos(\mu u - \theta \sigma^\alpha u^\alpha) \\ &\quad - \alpha \theta \sigma^\alpha u^{\alpha-1} \exp(-\sigma^\alpha u^\alpha) \sin(\mu u - \theta \sigma^\alpha u^\alpha), \end{aligned}$$

inserting these into (1.4) of Lemma 1.3, we get (2.7). For  $m_{1+\lambda}$ , we let  $\mu = 0$  in (2.7) and use change of variables theorem to obtain

$$m_{1+\lambda} = \frac{\lambda \sigma^{1+\lambda}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left\{ \int_0^\infty u^{-\frac{1+\lambda}{\alpha}} e^{-u} \cos \theta u du + \theta \int_0^\infty u^{-\frac{1+\lambda}{\alpha}} e^{-u} \sin \theta u du \right\}.$$

Now applying the formulae

$$\begin{aligned} \int_0^\infty x^{a-1} e^{-bx} \sin cx dx &= \frac{\Gamma(a)}{(b^2 + c^2)^{a/2}} \sin \left\{ a \arctan \frac{c}{b} \right\}, \quad [\operatorname{Re} a > -1, \operatorname{Re} b > |\operatorname{Im} c|], \\ \int_0^\infty x^{a-1} e^{-bx} \cos cx dx &= \frac{\Gamma(a)}{(b^2 + c^2)^{a/2}} \cos \left\{ a \arctan \frac{c}{b} \right\}, \quad [\operatorname{Re} a > 0, \operatorname{Re} b > |\operatorname{Im} c|], \end{aligned}$$

see (3.944,5) and (3.944,6) in [6], we get (2.6). Finally, letting  $\beta = 0$  and applying change of variables, the symmetric case is obtained.  $\square$

After some manipulation one can show that (2.8) coincides with (2.4) and (2.6) coincides with (2.5) for  $\gamma = 1 + \lambda$ . Figure 1 shows the fractional absolute moments with center  $\mu$ , computed numerically from the representation (2.7). We remark that even if this representation includes some integral expressions, it would be useful since most stable distributions have no explicit density functions.

**2.2. Pareto law.** Another heavy tailed distribution is the Pareto distribution which has density and ch.f. given by

$$\begin{aligned} f(x) &= \alpha(1+x)^{-\alpha-1}, \quad x > 0, \\ \varphi(t) &= \alpha \int_0^\infty e^{ity} (1+y)^{-\alpha-1} dy, \quad t \in \mathbb{R}, \end{aligned}$$

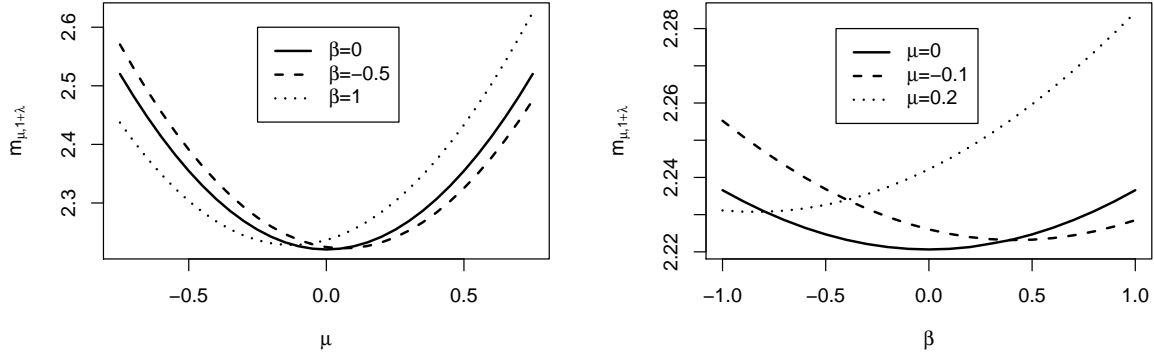


FIGURE 1. The moments  $m_{\mu,1+\lambda}$  of stable distribution with parameters  $\alpha = 1.8$ ,  $\beta \in [-1, 1]$ ,  $\delta = 0$  and  $\sigma = 1$ . We choose  $\lambda = 0.5$  and depict the dependence on  $\mu$  for three choices of  $\beta$  (left) and the dependence on  $\beta$  for three choices of  $\mu$  (right).

respectively, with real positive parameter  $\alpha > 0$ . This distribution belongs to ID distribution (see Remark 8.12 in [21]). The fractional absolute moment  $m_{1+\lambda}$  exists if and only if  $1 + \lambda < \alpha$ . Though the density function is explicit, we obtain  $m_{\mu,1+\lambda}$  from the fractional derivative of ch.f. Since

$$\begin{aligned} \operatorname{Im} e^{i\mu u} \varphi(-u) &= \alpha \int_0^\infty \sin\{(\mu - y)u\} (1 + y)^{-\alpha-1} dy, \\ \operatorname{Re} e^{i\mu u} \varphi'(-u) &= -\alpha \int_0^\infty y \sin\{(\mu - y)u\} (1 + y)^{-\alpha-1} dy, \end{aligned}$$

due to (1.4) of Lemma 1.3, we have, for  $1 < 1 + \lambda < \alpha$ ,

$$m_{\mu,1+\lambda} = \alpha \left\{ (\mu + 1)^{1+\lambda-\alpha} B(\alpha - 1 - \lambda, 2 + \lambda) + \frac{\mu^{2+\lambda}}{2 + \lambda} {}_2F_1(1, \alpha + 1, 3 + \lambda; -\mu) \right\},$$

where we use formulae (3.196,1), (3.196,2) and (3.761,4) in [6] together with Fubini's theorem. Here,  $B$  is the beta function and  ${}_2F_1$  is the Gauss hypergeometric function.

Although the following examples are not always in ID distributions, they are closely related and could be heavy tailed.

**2.3. Geometric stable law.** A geometric stable distribution has similar properties to the stable distribution. The ch.f. is given as

$$\varphi(t) = [1 + \sigma^\alpha |t|^\alpha \omega(t) - i\delta t]^{-1}, \quad t \in \mathbb{R},$$

where  $0 < \alpha < 2$  and  $\omega(t)$  is defined by (2.3). However, its density function has no analytical expression. The tail behavior is the same as that of stable distribution, see e.g. Kozubowski et al. [9].

**Lemma 2.3.** *Let  $X$  has a geometric stable distribution with  $\delta = 0$ . Then, for  $1 < 1 + \lambda < \alpha$  and  $\mu \in \mathbb{R}$ ,*

$$\begin{aligned} m_{\mu,1+\lambda} &= \frac{\lambda}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left[ \mu \int_0^\infty u^{-(1+\lambda)} \frac{(1 + \sigma^\alpha u^\alpha) \sin \mu u - \theta \sigma^\alpha u^\alpha \cos \mu u}{(1 + \sigma^\alpha u^\alpha)^2 + (\theta \sigma^\alpha u^\alpha)^2} du \right. \\ &\quad \left. - \alpha \sigma^\alpha \int_0^\infty u^{\alpha-\lambda-2} \frac{\cos \mu u + \theta \sin \mu u}{(1 + \sigma^\alpha u^\alpha)^2 + (\theta \sigma^\alpha u^\alpha)^2} du \right] \end{aligned}$$

$$+ 2\alpha\sigma^\alpha \int_0^\infty u^{\alpha-\lambda-2} \frac{\{(1+\sigma^\alpha u^\alpha) \cos \mu u + \theta \sigma^\alpha u^\alpha \sin \mu u\} \{1+\sigma^\alpha u^\alpha + \theta^2 \sigma^\alpha u^\alpha\}}{\{(1+\sigma^\alpha u^\alpha)^2 + (\theta \sigma^\alpha u^\alpha)^2\}^2} du \Big]$$

and

$$m_{1+\lambda} = \frac{\lambda \sigma^{1+\lambda}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \int_0^\infty v^{-\frac{1+\lambda}{\alpha}} \frac{(1+v)^2 + (\theta v)^2 + 2\theta^2 v}{\{(1+v)^2 + (\theta v)^2\}^2} dv,$$

where  $\theta = \beta \tan \frac{\pi\alpha}{2}$ .

*Proof.* The proof is an application of Lemma 1.3 to the following formulae for  $u > 0$ ,

$$\varphi(-u) = \frac{1 + \sigma^\alpha u^\alpha - i\theta \sigma^\alpha u^\alpha}{(1 + \sigma^\alpha u^\alpha)^2 + (\theta \sigma^\alpha u^\alpha)^2}$$

and

$$\begin{aligned} \varphi'(-u) = & -\frac{\alpha \sigma^\alpha u^{\alpha-1}}{(1 + \sigma^\alpha u^\alpha)^2 + (\theta \sigma^\alpha u^\alpha)^2} + \frac{2(1 + \sigma^\alpha u^\alpha)\{(1 + \sigma^\alpha u^\alpha)\alpha \sigma^\alpha u^{\alpha-1} + (\theta \sigma^\alpha u^\alpha)\alpha \theta \sigma^\alpha u^{\alpha-1}\}}{\{(1 + \sigma^\alpha u^\alpha)^2 + (\theta \sigma^\alpha u^\alpha)^2\}^2} \\ & + i \frac{\alpha \theta \sigma^\alpha u^{\alpha-1}}{(1 + \sigma^\alpha u^\alpha)^2 + (\theta \sigma^\alpha u^\alpha)^2} - i \frac{2\theta \sigma^\alpha u^\alpha \{(1 + \sigma^\alpha u^\alpha)\alpha \sigma^\alpha u^{\alpha-1} + (\theta \sigma^\alpha u^\alpha)\alpha \theta \sigma^\alpha u^{\alpha-1}\}}{\{(1 + \sigma^\alpha u^\alpha)^2 + (\theta \sigma^\alpha u^\alpha)^2\}^2}. \end{aligned}$$

□

If we put  $\theta = 0$  the results coincide with the standard Linnik law case.

**2.4. Linnik law.** We consider a version of Linnik distribution given by Linnik [14]. Its density function is not explicit, while its ch.f. has the form

$$\varphi(t) = (1 + \sigma^\alpha |t|^\alpha)^{-\beta}, \quad t \in \mathbb{R},$$

where  $0 < \alpha \leq 2$  is the stability parameter,  $\sigma > 0$  is the scale parameter and  $\beta > 0$ . By the method of fractional derivative, we recover the result of Lin [13] as

$$m_{1+\lambda} = \frac{\lambda \beta \sigma^{1+\lambda}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} B\left(1 - \frac{1+\lambda}{\alpha}, \beta + \frac{1+\lambda}{\alpha}\right),$$

where  $1 < 1+\lambda < \alpha$ . The fractional absolute moment of order  $1 < 1+\lambda < \alpha$  with center  $\mu \in \mathbb{R}$  is

$$(2.10) \quad m_{\mu,1+\lambda} = \frac{\lambda \sigma^{1+\lambda}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left\{ \frac{\mu}{\sigma} \int_0^\infty \frac{u^{-(1+\lambda)} \sin \frac{\mu u}{\sigma}}{(1+u^\alpha)^\beta} du + \alpha \beta \int_0^\infty \frac{u^{\alpha-\lambda-2} \cos \frac{\mu u}{\sigma}}{(1+u^\alpha)^{\beta+1}} du \right\}.$$

For  $\beta = 1$  these equations coincide with those in Lemma 2.3 for  $\theta = 0$ .

**2.5. Combination of stable law and Linnik law.** Since

$$\lim_{\beta \rightarrow \infty} (1 + \sigma^\alpha |t|^\alpha / \beta)^{-\beta} = e^{-\sigma^\alpha |t|^\alpha}, \quad t \in \mathbb{R}, \quad 0 < \alpha \leq 2,$$

a symmetric stable distribution is a limit of Linnik-type distributions. We consider their combination keeping both exponents  $\alpha$  to be identical. Let  $X$  be a symmetric stable with ch.f.  $\varphi(t) = e^{-|t|^\alpha}$  and let  $Y$  be a Linnik-type distribution with ch.f.  $\varphi(t) = (1 + |t|^\alpha / \beta)^{-\beta}$ . Then we may express  $E[|X - Y|^{1+\lambda}]$  by taking expectation of (2.10) with  $\mu$  replaced by  $X$  and  $\sigma = \beta^{-1/\alpha}$ . As the result we obtain

$$\begin{aligned} E[|X - Y|^{1+\lambda}] &= \frac{\lambda \beta^{1-\frac{1+\lambda}{\alpha}}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left\{ \int_0^\infty u^{-\frac{1+\lambda}{\alpha}} (1+u)^{-\beta} e^{-\beta u} du + \int_0^\infty u^{-\frac{1+\lambda}{\alpha}} (1+u)^{-\beta-1} e^{-\beta u} du \right\} \\ &= \frac{\lambda \beta^{1-\frac{1+\lambda}{\alpha}} \Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left\{ U\left(1 - \frac{1+\lambda}{\alpha}, 2 - \beta - \frac{1+\lambda}{\alpha}; \beta\right) + U\left(1 - \frac{1+\lambda}{\alpha}, 1 - \beta - \frac{1+\lambda}{\alpha}; \beta\right) \right\}, \end{aligned}$$

where  $U$  is the confluent hypergeometric function (9.210,2) of [6].

**2.6. Subordinator.** For practical purpose, it is desirable to express  $m_{1+\lambda}$  through Lévy measure  $\nu$  since ID distributions without Gaussian part are completely characterized by centering parameter  $\delta$  and Lévy measure. However, in the light of (2.1), such expressions seem to be too formal and too complicated, thus they seem to be not very useful. Here, we confine our interest to some well-known distributions. However, for small classes of ID distributions general expressions of  $m_{1+\lambda}$  by  $\nu$  are worth considering. We pick out the class of subordinator (positive valued ID distributions) and that of compound Poisson distributions, the latter is treated in Section 3.

For subordinator, we apply Lemma 1.1 and obtain a relatively simple expression. The LP transform of a subordinator can be found in [21, Theorem 30.1].

**Proposition 2.4.** *Let  $X$  be a positive valued ID random variable with shift parameter  $\delta \geq 0$  and Lévy measure  $\nu$  such that  $\int_{(0,\infty)} (1 \wedge |s|) \nu(ds) < \infty$ . The LP transform is given by*

$$\phi(t) = e^{\Psi(-t)}, \quad t \geq 0,$$

where

$$\Psi(t) = \delta t + \int_{(0,\infty)} (e^{st} - 1) \nu(ds).$$

Then it follows that

$$m_{1+\lambda} = \frac{\lambda}{\Gamma(1-\lambda)} \left\{ \mu \int_0^\infty \frac{1 - e^{\Psi(-u)}}{u^{1+\lambda}} du + \int_0^\infty s \nu(ds) \int_0^\infty \frac{1 - e^{-us} e^{\Psi(-u)}}{u^{1+\lambda}} du \right\}.$$

### 3. COMPOUND POISSON DISTRIBUTION

Among ID distributions we focus on compound Poisson (CP for short) distribution which can easily manage the tail behavior by assuming a heavy tailed jump distribution. However, since most distributions do not have explicit representations, we rely on the ch.f. or the LP transform for calculating fractional moments. Let  $c$  be the intensity parameter of underlying Poisson distribution and  $\nu$  jump measure. The CP distribution has the following ch.f.

$$(3.1) \quad \varphi(t) = \exp \left\{ c \left( \int (e^{itx} - 1) \nu(dx) \right) \right\} := \exp \{ c(\varphi_J(t) - 1) \}, \quad t \in \mathbb{R},$$

where  $\varphi_J(t) := \int e^{itx} \nu(dx)$  is the ch.f. of jump distribution. If the jump distribution has positive support, we obtain the LP transform

$$\phi(t) := \exp \{ c(\phi_J(t) - 1) \}, \quad t \geq 0,$$

where  $\phi_J(t) := \int e^{-tx} \nu(dx)$ . The fractional absolute moments are expressed in the following lemma. The proof is just an application of Lemma 1.1 and Lemma 1.3 and we omit it.

**Lemma 3.1.** *Let  $\varphi(t)$  be the ch.f. of CP given by (3.1), then we have the following form for fractional  $\mu$ -centered moments of order  $1 < 1 + \lambda < 2$ ,*

$$\begin{aligned} m_{\mu,1+\lambda} &= \frac{\lambda}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left[ \mu \int_0^\infty u^{-(1+\lambda)} \sin\{\mu u + c \operatorname{Im}(\varphi_J(-u))\} \exp\{c(\operatorname{Re}(\varphi_J(-u)) - 1)\} du \right. \\ &\quad + c \int_0^\infty u^{-(1+\lambda)} \operatorname{Re}(\varphi_J'(-u)) \cos\{\mu u + c \operatorname{Im}(\varphi_J(-u))\} \exp\{c(\operatorname{Re}(\varphi_J(-u)) - 1)\} du \\ &\quad \left. - c \int_0^\infty u^{-(1+\lambda)} \operatorname{Im}(\varphi_J'(-u)) \sin\{\mu u + c \operatorname{Im}(\varphi_J(-u))\} \exp\{c(\operatorname{Re}(\varphi_J(-u)) - 1)\} du \right]. \end{aligned}$$

If the jump distribution is symmetric, i.e.  $\operatorname{Im} \varphi_J(u) = 0$ , we have

$$m_{\mu,1+\lambda} = \frac{\lambda}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left[ \mu \int_0^\infty u^{-(1+\lambda)} \sin(\mu u) \exp\{c(\varphi_J(-u) - 1)\} du \right]$$

$$+ c \int_0^\infty u^{-(1+\lambda)} \varphi'_J(-u) \cos(\mu u) \exp\{c(\varphi_J(-u) - 1)\} du \Big]$$

and moreover

$$m_{1+\lambda} = \frac{\lambda c}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \int_0^\infty u^{-(1+\lambda)} \varphi'_J(-u) \exp\{c(\varphi_J(-u) - 1)\} du.$$

If the jump distribution has positive support, we have

$$m_{1+\lambda} = \frac{\lambda c}{\Gamma(1-\lambda)} \int_0^\infty u^{-(1+\lambda)} \{\phi'_J(u) \exp\{c(\phi_J(u) - 1)\} - \phi'_J(0)\} du.$$

In what follows, we will examine jumps given by well known distributions, which are not always heavy tailed, and try to obtain analytical expressions. Since they require a lot of numerical integrals and special functions we do not write complete calculation processes and just mention the key steps of derivation.

**3.1. exponential jump.** The LP transform of the exponential distribution with parameter  $\beta$ , i.e. with density function  $\frac{1}{\beta}e^{x/\beta}$ ,  $x \geq 0$ , is  $\phi_J(u) = 1/(1 + \beta u)$ ,  $u \geq 0$ . Then due to Lemma 3.1 fractional absolute moments for  $0 < \lambda < 1$  are given by

$$\begin{aligned} m_{1+\lambda} &= \frac{\lambda c \beta}{\Gamma(1-\lambda)} \int_0^\infty \frac{(1 + \beta u)^2 - \exp\{c(\frac{1}{1+\beta u} - 1)\}}{u^{1+\lambda}(1 + \beta u)^2} du \\ &= c\beta^{1+\lambda}\Gamma(2+\lambda) \left\{ {}_1F_1(1-\lambda; 2; -c) + \frac{c}{2} {}_1F_1(1-\lambda; 3; -c) \right\}, \end{aligned}$$

where  ${}_1F_1$  is the confluent hypergeometric function (9.210,1) of [6] and we use (3.383,1) and (3.191,3) of [6].

**3.2. symmetric stable jump.** Recall that the ch.f. is  $\phi_J(u) = e^{-|u|^\alpha}$  with  $1 < \alpha < 2$  and thus we apply Lemma 3.1 with  $1 < 1 + \lambda < \alpha$  to obtain the following series representation,

$$\begin{aligned} m_{1+\lambda} &= \frac{\lambda \alpha c}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \int_0^\infty u^{\alpha-\lambda-2} e^{-u^\alpha} \exp\{c(e^{-u^\alpha} - 1)\} du \\ &= \frac{\lambda c}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \int_0^\infty v^{-(1+\lambda)/\alpha} e^{-v} \exp\{c(e^{-v} - 1)\} dv \\ &= \frac{\lambda c}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} e^{-c} \sum_{n=0}^\infty c^n \frac{\Gamma(1 - (1+\lambda)/\alpha)}{n!(n+1)^{1-(1+\lambda)/\alpha}}. \end{aligned}$$

Again by Lemma 3.1, shifted fractional moments  $E[|X - \mu|^{1+\lambda}]$  with  $1 < 1 + \lambda < \alpha$  are obtained as

$$\begin{aligned} m_{\mu,1+\lambda} &= \frac{\lambda}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left[ \mu \int_0^\infty u^{-(1+\lambda)} \sin(\mu u) \exp\{c(e^{-u^\alpha} - 1)\} du \right. \\ &\quad \left. + c\alpha \int_0^\infty u^{\alpha-\lambda-2} \cos(\mu u) e^{-u^\alpha} \exp\{c(e^{-u^\alpha} - 1)\} du \right]. \end{aligned}$$

**3.3. Linnik distribution jump.** Let  $\varphi_J$  be the ch.f. of Linnik distribution with parameters  $\alpha > 1$ ,  $\beta > 0$  and  $\sigma = 1$ . Since

$$\varphi'_J(-u) = \alpha\beta(1 + u^\alpha)^{-\beta-1}u^{\alpha-1}, \quad u > 0,$$

from Lemma 3.1 and change of variables formula ( $v = (1 + u^\alpha)^{-\beta}$ ) it follows that

$$m_{1+\lambda} = \frac{\lambda c e^{-c}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \int_0^1 v^{\frac{1+\lambda}{\alpha\beta}} (1 - v^{\frac{1}{\beta}})^{-\frac{1+\lambda}{\alpha}} e^{cv} dv$$

for  $1 < 1 + \lambda < \alpha$ . If  $\beta = 1$ , the jump distribution is the symmetric geometric stable distribution and we have

$$m_{1+\lambda} = \frac{\lambda c e^{-c}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} B(1 - \frac{1+\lambda}{\alpha}, 1 + \frac{1+\lambda}{\alpha}) {}_1F_1(1 + \frac{1+\lambda}{\alpha}; 2; c),$$

where we use (3.383,1) in [6].

**3.4. deterministic jump of size 1 (simple Poisson).** Substituting its LP transform  $\phi_J(u) = e^{-u}$  into the expression in Lemma 3.1, we have

$$m_{1+\lambda} = \frac{\lambda c e^{-c}}{\Gamma(1-\lambda)} \int_0^\infty \frac{e^c - e^{-u} e^{ce^{-u}}}{u^{1+\lambda}} du,$$

which is rewritten by the Taylor expansion as

$$m_{1+\lambda} = \frac{\lambda c e^{-c}}{\Gamma(1-\lambda)} \sum_{k=0}^\infty \frac{c^k}{k!} \int_0^\infty \frac{1 - e^{-(k+1)u}}{u^{1+\lambda}} du = e^{-c} \sum_{k=0}^\infty \frac{k^{1+\lambda} c^k}{k!},$$

where the final expression can be directly obtained from probability mass function.

**Remark 3.2.** *If the jump distribution has reproductive property, i.e. it is convolution-closed, we have another method for determining the fractional absolute moments. Write the CP random variable as  $S_N = \sum_{j=1}^N X_j$ , where  $N$  has the Poisson distribution with parameter  $c$  and  $(X_j)$  is an iid sequence such that  $X_1$  has reproduction property. Denote the ch.f. of  $k$ th convolution of  $X_1$  by  $\varphi_k(u)$ , then under suitable conditions we have*

$$\begin{aligned} m_{1+\lambda} &= \frac{\lambda}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} E \left[ \operatorname{Re} \int_0^\infty \frac{\varphi'_N(-u)}{u^{1+\lambda}} du \right] \\ &= \frac{\lambda}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \sum_{k=0}^\infty \frac{c^k}{k!} e^{-c} \left[ \operatorname{Re} \int_0^\infty \frac{\varphi'_k(-u)}{u^{1+\lambda}} du \right]. \end{aligned}$$

In case of the LP transform we denote that of  $k$ th convolution of  $X_1$  by  $\phi_k(u)$ ,  $u \geq 0$ , and from Lemma 1.1 we obtain

$$m_{1+\lambda} = \frac{\lambda}{\Gamma(1-\lambda)} E \left[ \int_0^\infty \frac{\phi'_N(u) - \phi'_N(0+)}{u^{1+\lambda}} du \right] = \frac{\lambda e^{-c}}{\Gamma(1-\lambda)} \sum_{k=0}^\infty \frac{c^k}{k!} \int_0^\infty \frac{\phi'_k(u) - \phi'_k(0+)}{u^{1+\lambda}} du.$$

## 4. APPLICATIONS

**4.1. Evaluation of conditional expectation for stable law.** Conditional expectations of stable random vectors have been intensively investigated in [7] and [19], since stable laws are often thought as natural generalization of the Gaussian random vector for which the minimizer of the mean squared error given some components of the vector is the conditional expectation. However, their evaluations have not been examined enough. In what follows, we evaluate the goodness of several predictors given by conditional expectations through their fractional moments.

Firstly, we consider general results for a bivariate stable random vector with ch.f.

$$\varphi(t_1, t_2) := E[e^{i(t_1 X_1 + t_2 X_2)}], \quad (t_1, t_2) \in \mathbb{R}^2,$$

which can be written as

$$(4.1) \quad \varphi(t_1, t_2) = \exp \left\{ - \int_{\mathbb{S}^1} |t_1 s_1 + t_2 s_2|^\alpha \left( 1 - i \tan \frac{\pi\alpha}{2} \operatorname{sign}(t_1 s_1 + t_2 s_2) \right) \Gamma(ds) + i(t_1 \delta_1 + t_2 \delta_2) \right\},$$

where  $\Gamma$  is a finite measure on the unit sphere  $\mathbb{S}^1$ , called spectral measure, and we let  $\alpha > 1$ , see [20, Theorem 2.3.1]. Our aim is to linearly approximate  $X_2$  by  $X_1$  and evaluate the fractional

error of order  $1 < \gamma < 2$ . The situation includes various settings, e.g. if stable random vectors are symmetric, i.e.

$$\varphi(t_1, t_2) = \exp \left\{ - \int_{\mathbb{S}^1} |t_1 s_1 + t_2 s_2|^\alpha \Gamma(ds) \right\},$$

then it is proved that  $E[X_2 | X_1] = cX_1$  with some constant  $c$ , see [20, Theorem 4.1.2] or [19, Theorem 3.1]. For general case we refer to [7, Theorem 3.1]. For convenience, we assume  $(\delta_1, \delta_2) = \mathbf{0}$ , the general result for  $(\delta_1, \delta_2) \neq \mathbf{0}$  can be obtained in the same manner.

**Proposition 4.1.** *Let  $(X_1, X_2)$  be a bivariate stable random vector defined by (4.1) such that  $(\delta_1, \delta_2) = \mathbf{0}$ . Then for any constant  $c$  and  $1 < 1 + \lambda < \alpha$ , it follows that*

$$\begin{aligned} E[|X_2 - cX_1|^{1+\lambda}] &= \frac{\lambda \Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2}) \Gamma(1 - \lambda)} \sigma_0^{1+\lambda} (1 + \theta_0^2)^{\frac{1+\lambda}{2\alpha} - \frac{1}{2}} \\ &\quad \times \left[ \cos \left\{ \left(1 - \frac{1+\lambda}{\alpha}\right) \arctan \theta_0 \right\} + \theta_0 \sin \left\{ \left(1 - \frac{1+\lambda}{\alpha}\right) \arctan \theta_0 \right\} \right], \end{aligned}$$

where

$$(4.2) \quad \sigma_0 = \left( \int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(ds) \right)^{1/\alpha}, \quad \beta_0 = \frac{\int_{\mathbb{S}^1} \text{sign}(s_2 - cs_1) |s_2 - cs_1|^\alpha \Gamma(ds)}{\int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(ds)}$$

and  $\theta_0 = \beta_0 \tan \frac{\pi\alpha}{2}$ . In symmetric case, we have

$$E[|X_2 - cX_1|^{1+\lambda}] = \frac{\lambda \Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2}) \Gamma(1 - \lambda)} \left( \int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(ds) \right)^{\frac{1+\lambda}{\alpha}}.$$

*Proof.* The fractional derivative of the ch.f. of  $X_2 - cX_1$  is calculated. We put  $t_1 = -cu$  and  $t_2 = u$  in (4.1), then we regard it as a function of  $u$ ,

$$\begin{aligned} E[e^{iu(X_2 - cX_1)}] &= \exp \left\{ - |u|^\alpha \int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(ds) \left( 1 - i \tan \frac{\pi\alpha}{2} \text{sign}(u) \frac{\int_{\mathbb{S}^1} \text{sign}(s_2 - cs_1) |s_2 - cs_1|^\alpha \Gamma(ds)}{\int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(ds)} \right) \right\}. \end{aligned}$$

In view of (2.2), this is the ch.f. of one-dimensional  $\alpha$ -stable distribution with parameters  $(\beta, \sigma, \delta) = (\beta_0, \sigma_0, 0)$ . Hence, we apply Proposition 2.2 to obtain the result.  $\square$

**Examples.** As examples we consider predictions for two bivariate stable random vectors and one stable process. First we treat a bivariate stable random vector considered by [16, p. 183] such that ch.f. of  $(X_1, X_2)$  satisfies for  $|a| < 1$ ,  $\alpha \neq 1$ ,

$$\begin{aligned} \varphi(t_1, t_2) &= E[e^{i(t_1 X_1 + t_2 X_2)}] = \exp \left\{ - \sigma^\alpha |t_2|^\alpha \left( 1 + i\beta \tan \frac{\pi\alpha}{2} \text{sign}(t_2) \right) \right. \\ &\quad \left. - \frac{\sigma^\alpha}{1 - |a|^\alpha} |t_1 + at_2|^\alpha \left( 1 + i\beta \tan \frac{\pi\alpha}{2} \frac{1 - |a|^\alpha}{1 - \text{sign}(a)|a|^\alpha} \text{sign}(t_1 + at_2) \right) \right\}. \end{aligned}$$

The conditional ch.f. is

$$\varphi_{X_1=x}(u) := E[e^{iuX_2} | X_1 = x] = \exp \left\{ iaxu - \sigma^\alpha |u|^\alpha \left( 1 + i\beta \tan \frac{\pi\alpha}{2} \text{sign}(u) \right) \right\}$$

and hence for  $1 < \alpha < 2$ ,  $E[X_2 | X_1 = x] = ax$ . The support of spectral measure  $\Gamma$  consists of four points in  $\mathbb{S}^1$ ,

$$\begin{aligned} \Gamma(0, \pm 1) &= \frac{1}{2} \sigma^\alpha (1 \pm \beta), \\ \Gamma\left(\pm \frac{1}{\sqrt{1+a^2}}, \pm \frac{a}{\sqrt{1+a^2}}\right) &= \frac{1}{2} \frac{\sigma^\alpha}{1 - |a|^\alpha} (1 + a^2)^{\frac{\alpha}{2}} \left( 1 \pm \beta \frac{1 - |a|^\alpha}{1 - \text{sign}(a)|a|^\alpha} \right). \end{aligned}$$

Hence,

$$\int_{\mathbb{S}^1} |s_2 - cs_1|^\alpha \Gamma(ds) = \sigma^\alpha \left(1 + \frac{|a - c|^\alpha}{1 - |a|^\alpha}\right),$$

$$\int_{\mathbb{S}^1} \text{sign}(s_2 - cs_1) |s_2 - cs_1|^\alpha \Gamma(ds) = \beta \sigma^\alpha \left(1 + \frac{\text{sign}(a - c) |a - c|^\alpha}{1 - \text{sign}(a) |a|^\alpha}\right).$$

Substitution of these relations into (4.2) yields  $\sigma_0$  and  $\beta_0$  that can be used for calculating the fractional absolute prediction error by Proposition 4.1.

Another example is the prediction for sub-Gaussian random vector. Let  $0 < \alpha < 2$ ,  $|\gamma| \leq 1$ , and let  $(G_1, G_2)$  be zero mean Gaussian random vector with covariance matrix

$$(4.3) \quad \Sigma = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}.$$

Let  $A$  be a positive  $\alpha/2$ -stable random variable, given by the LP transform

$$E[e^{-uA}] = e^{-u^{\alpha/2}}, \quad u > 0,$$

such that it is independent of  $(G_1, G_2)$ . The vector  $(X_1, X_2) = (A^{1/2}G_1, A^{1/2}G_2)$  is called a sub-Gaussian symmetric  $\alpha$ -stable random vector. In [19],  $E[X_2 | X_1] = \gamma X_1$  is shown. Since we have the ch.f.

$$\begin{aligned} E[e^{iu(X_2 - \gamma X_1)}] &= E[E[e^{iu(A^{1/2}G_2 - \gamma A^{1/2}G_1)} | A]] = E[\exp\{-\frac{u^2}{2}A(-\gamma, 1)\Sigma(-\gamma, 1)'\}] \\ &= E[\exp\{-\frac{u^2(1 - \gamma^2)}{2}A\}] = \exp\{-(\frac{1 - \gamma^2}{2})^{\alpha/2}u^\alpha\}, \end{aligned}$$

due to the fractional moment (2.8), we get the fractional error

$$E[|X_2 - E[X_2 | X_1]|^{1+\lambda}] = \frac{\lambda \Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2})\Gamma(1 - \lambda)} \left(\frac{1 - \gamma^2}{2}\right)^{\frac{1+\lambda}{2}},$$

where  $1 < 1 + \lambda < \alpha$ . If  $X_2$  is predicted by a linear function  $cX_1$ , in a similar manner, we obtain

$$E[e^{i(X_2 - cX_1)}] = \exp\left\{-\left(\frac{1 - 2\gamma c + c^2}{2}\right)^{\alpha/2}u^\alpha\right\},$$

which yields

$$E[|X_2 - cX_1|^{1+\lambda}] = \frac{\lambda \Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2})\Gamma(1 - \lambda)} \left(\frac{1 - 2\gamma c + c^2}{2}\right)^{\frac{1+\lambda}{2}}.$$

Alternatively, we could use the spectral measure of sub-Gaussian random vector given in [20, Proposition 2.5.8]. Since it is given in a closed form, we obtain the fractional error directly from ch.f. here.

Next we examine the prediction of the  $\alpha$ -stable OU process with  $0 < \alpha < 2$  and  $\gamma > 0$  given by

$$X_t = e^{-\gamma t}X_0 + \int_0^t e^{-\gamma(t-s)} dZ_s, \quad t > 0,$$

where  $\{Z_t\}_{t \in \mathbb{R}}$  is the symmetric  $\alpha$ -stable motion. We set  $X_0 = \int_{-\infty}^0 e^{\gamma s} dZ_s$  to obtain the stationary version, see [20, Example 3.6.3] for its definition. Then, the conditional ch.f. of  $X_t$  given  $X_0$  is

$$\begin{aligned} \varphi_{X_0}(u) &= E[e^{iuX_t} | X_0] = \exp\{iue^{-\gamma t}X_0\} \exp\left\{-\int_0^t |ue^{-\gamma(t-s)}|^\alpha ds\right\} \\ &= \exp\left\{iue^{-\gamma t}X_0 - \frac{1 - e^{-\alpha\gamma t}}{\alpha\gamma}|u|^\alpha\right\}, \end{aligned}$$

which yields  $E[X_t | X_0] = e^{-\gamma t} X_0$  for  $\alpha > 1$ . Since the mean squared error of the prediction is not available, we use the fractional absolute moment of order  $1 < 1 + \lambda < \alpha$ . More generally, we measure the error of a linear approximation  $cX_0$  with  $c \in \mathbb{R}$ .

**Proposition 4.2.** *Let  $X_t$  be an  $\alpha$ -stable OU-process driven by the symmetric stable motion with the location parameter  $\delta = 0$ . Then for  $1 < 1 + \lambda < \alpha$ ,*

$$E[|X_t - cX_0|^{1+\lambda}] = \frac{\lambda\Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left\{ \frac{1 - e^{-\alpha\gamma t} + (c - e^{-\gamma t})^\alpha}{\alpha\gamma} \right\}^{\frac{1+\lambda}{\alpha}}$$

and hence putting  $c = e^{-\gamma t}$ , we obtain

$$E[|X_t - E[X_t | X_0]|^{1+\lambda}] = \frac{\lambda\Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left( \frac{1 - e^{-\alpha\gamma t}}{\alpha\gamma} \right)^{\frac{1+\lambda}{\alpha}}.$$

*Proof.* Since

$$E[e^{iu(X_t - cX_0)} | X_0] = \exp\{iu(e^{-\gamma t} - c)X_0\} \exp\left\{-\frac{1}{\alpha\gamma}(1 - e^{-\alpha\gamma t})|u|^\alpha\right\}, \quad u \in \mathbb{R},$$

we may use formula (2.9) in Proposition 2.2 with  $\mu = (c - e^{-\gamma t})X_0$  and  $\sigma^\alpha = \frac{1}{\alpha\gamma}(1 - e^{-\alpha\gamma t})$ . Consequently,

$$\begin{aligned} E[|X_t - cX_0|^{1+\lambda} | X_0] &= \frac{\lambda\sigma^{1+\lambda}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left\{ \frac{(c - e^{-\gamma t})X_0}{\sigma} \int_0^\infty v^{-(1+\lambda)} \sin\left[\frac{(c - e^{-\gamma t})X_0}{\sigma}v\right] e^{-v^\alpha} dv \right. \\ &\quad \left. + \alpha \int_0^\infty v^{\alpha-\lambda-2} \cos\left[\frac{(c - e^{-\gamma t})X_0}{\sigma}v\right] e^{-v^\alpha} dv \right\}. \end{aligned}$$

After taking expectation w.r.t.  $X_0$  and applying Fubini's theorem, we use

$$\begin{aligned} E \cos\left[\frac{(c - e^{-\gamma t})X_0}{\sigma}v\right] &= \exp\left\{-\frac{1}{\alpha\gamma}\left(\frac{c - e^{-\gamma t}}{\sigma}\right)^\alpha v^\alpha\right\}, \\ E \frac{(c - e^{-\gamma t})X_0}{\sigma} \sin\left[\frac{(c - e^{-\gamma t})X_0}{\sigma}v\right] &= \frac{v^{\alpha-1}}{\gamma} \left(\frac{c - e^{-\gamma t}}{\sigma}\right)^\alpha \exp\left\{-\frac{1}{\alpha\gamma}\left(\frac{c - e^{-\gamma t}}{\sigma}\right)^\alpha v^\alpha\right\}, \end{aligned}$$

to get

$$\begin{aligned} E[|X_t - cX_0|^{1+\lambda}] &= \frac{\lambda\sigma^{1+\lambda}}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} \left\{ 1 + \frac{1}{\alpha\gamma} \left(\frac{c - e^{-\gamma t}}{\sigma}\right)^\alpha \right\} \\ &\quad \times \int_0^\infty u^{-\frac{1+\lambda}{\alpha}} \exp\left\{-\left\{1 + \frac{1}{\alpha\gamma} \left(\frac{c - e^{-\gamma t}}{\sigma}\right)^\alpha\right\}u\right\} du, \end{aligned}$$

where we apply change of variables several times. Then the result is implied by  $\sigma^\alpha = \frac{1}{\alpha\gamma}(1 - e^{-\alpha\gamma t})$  and definition of the gamma function.  $\square$

Since the finite dimensional distribution of the  $\alpha$ -stable OU process is a multivariate stable, we may use Proposition 4.1 similarly as before. The spectral measure is given in [19, Example 3.6.4].

**4.2. Evaluation of conditional expectation for Linnik law.** Let  $(X_1, X_2)$  be a bivariate Linnik distribution with ch.f.

$$\varphi(t_1, t_2) = \{1 + (\mathbf{t}'\Sigma\mathbf{t})^{\alpha/2}\}^{-\beta}, \quad \mathbf{t}' = (t_1, t_2) \in \mathbb{R}^2,$$

where  $0 < \alpha \leq 2$ ,  $\beta > 0$  and  $\Sigma$  is given by (4.3), see e.g. [12].

**Proposition 4.3.** *Let  $(X_1, X_2)$  be a bivariate Linnik random vector. Then  $E[X_2 | X_1] = \gamma X_1$  and for any  $c \in \mathbb{R}$  and  $1 < \alpha < 2$  it follows that*

$$E[|X_2 - cX_1|^{1+\lambda}] = \frac{\lambda\beta}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} |c^2 - 2\gamma c + 1|^{\frac{1+\lambda}{2}} B(1 - \frac{1+\lambda}{\alpha}, \beta + \frac{1+\lambda}{\alpha})$$

and therefore

$$E[|X_2 - E[X_2 | X_1]|^{1+\lambda}] = \frac{\lambda\beta}{\sin(\frac{\lambda\pi}{2})\Gamma(1-\lambda)} (1 - \gamma^2)^{\frac{1+\lambda}{2}} B(1 - \frac{1+\lambda}{\alpha}, \beta + \frac{1+\lambda}{\alpha}).$$

*Proof.* To obtain  $E[X_2 | X_1]$ , we use the decomposition by [5] of univariate Linnik law, which is also applicable in our bivariate case. Let  $(Y_1, Y_2)$  be a sub-Gaussian random vector with ch.f.

$$\varphi(t_1, t_2) = e^{-(t' \Sigma t)^{\alpha/2}}, \quad \mathbf{t} = (t_1, t_2)'$$

and let  $Z$  be an independent random variable with density

$$f(x) = \frac{e^{-x^{1/\beta}}}{\Gamma(1+\beta)}, \quad x > 0.$$

Then we observe that  $(X_1, X_2) \stackrel{d}{=} (Y_1 Z^{1/\alpha\beta}, Y_2 Z^{1/\alpha\beta})$ , which leads to

$$\begin{aligned} E[X_2 | X_1] &\stackrel{d}{=} E[E[Y_2 Z^{1/\alpha\beta} | Y_1, Z^{1/\alpha\beta}] | Y_1 Z^{1/\alpha\beta}] \\ &= E[Z^{1/\alpha\beta} E[Y_2 | Y_1] | Y_1 Z^{1/\alpha\beta}] = \gamma Y_1 Z^{1/\alpha\beta} \stackrel{d}{=} \gamma X_1, \end{aligned}$$

where the conditional expectation of the sub-Gaussian random vector is used. Now put  $t_1 = -cu$  and  $t_2 = u$  in  $\varphi(t_1, t_2)$  to obtain

$$E[e^{iu(X_2 - cX_1)}] = \{1 + (c^2 - 2\gamma c + 1)^{\alpha/2} |u|^\alpha\}^{-\beta}$$

and we conclude our result from Subsection 2.4. □

**4.3. Estimation errors of regression model.** We consider the basic regression model

$$Y_j = \theta_0 + x_j \theta_1 + \varepsilon_j, \quad j = 1, 2, \dots, n,$$

where  $(\varepsilon_i)$  is an iid sequence of symmetric random variables. It is well-known that the least squares estimator  $(\hat{\theta}_0, \hat{\theta}_1)$ , which is the best linear unbiased estimator if the  $\varepsilon_j$  follow Gaussian distribution, has the form

$$\hat{\theta}_0 = \bar{Y} - \hat{\theta}_1 \bar{x}, \quad \hat{\theta}_1 = \frac{\sum_{j=1}^n (x_j - \bar{x})(Y_j - \bar{Y})}{\sum_{j=1}^n (x_j - \bar{x})^2},$$

where  $\bar{x} = \frac{1}{n} \sum_{j=1}^n x_j$  and  $\bar{Y} = \frac{1}{n} \sum_{j=1}^n Y_j$ . For our purpose, it will be convenient to rewrite this as

$$\begin{aligned} \hat{\theta}_0 &= \theta_0 - \sum_{i=1}^n \frac{(x_i - \bar{x})\bar{x} - \sum_{j=1}^n (x_j - \bar{x})^2/n}{\sum_{j=1}^n (x_j - \bar{x})^2} \varepsilon_i, \\ \hat{\theta}_1 &= \theta_1 + \sum_{i=1}^n \frac{x_i - \bar{x}}{\sum_{j=1}^n (x_j - \bar{x})^2} \varepsilon_i. \end{aligned}$$

We express the fractional errors for the case of  $\alpha$ -stable noise distribution. In a similar manner, it would be possible to calculate the fractional errors for other regression-type estimators, e.g. [1], and compare the goodness of estimators.

If  $\varepsilon_1$  is a standard symmetric stable random variable with parameters  $\delta = 0$ ,  $\sigma = 1$  and  $\alpha > 1$ , then the characteristic functions of estimation errors are

$$E[e^{iu(\hat{\theta}_k - \theta_k)}] = e^{-\sigma_k^\alpha |u|^\alpha}, \quad k = 0, 1,$$

where

$$\sigma_0^\alpha = \sum_{i=1}^n \left( \frac{|(x_i - \bar{x})\bar{x} - \sum_{j=1}^n (x_j - \bar{x})^2/n|}{\sum_{j=1}^n (x_j - \bar{x})^2} \right)^\alpha \quad \text{and} \quad \sigma_1^\alpha = \sum_{i=1}^n \left( \frac{|x_i - \bar{x}|}{\sum_{j=1}^n (x_j - \bar{x})^2} \right)^\alpha.$$

This together with (2.8) yields, for  $1 < 1 + \lambda < \alpha$ ,

$$E[|\hat{\theta}_k - \theta_k|^{1+\lambda}] = \frac{\lambda \Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2}) \Gamma(1 - \lambda)} \sigma_k^{1+\lambda}, \quad k = 0, 1.$$

Interestingly, if  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  is an elliptically contoured stable random vector with ch.f.  $E[e^{i\mathbf{u}'\varepsilon}] = e^{-|\mathbf{u}'\mathbf{I}\mathbf{u}|^{\alpha/2}}$  for  $\mathbf{u}' \in \mathbb{R}^n$ , where  $\mathbf{I}$  is  $n \times n$  identity matrix, then we obtain closer results to Gaussian case. Namely, for  $1 < 1 + \lambda < \alpha$ ,

$$E[|\hat{\theta}_k - \theta_0|^{1+\lambda}] = \frac{\lambda \Gamma(1 - \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2}) \Gamma(1 - \lambda)} \bar{\sigma}_k^{1+\lambda}, \quad k = 0, 1,$$

where

$$\bar{\sigma}_0 = \left( \frac{\bar{x}^2}{\sum_{j=1}^n (x_j - \bar{x})^2} + \frac{1}{n} \right)^{1/2} \quad \text{and} \quad \bar{\sigma}_1 = \frac{1}{(\sum_{j=1}^n (x_j - \bar{x})^2)^{1/2}}.$$

Moreover, if  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$  is a multivariate Linnik random vector with ch.f.  $E[e^{i\mathbf{u}'\varepsilon}] = \{1 + (\mathbf{u}'\mathbf{I}\mathbf{u})^{\alpha/2}\}^{-\beta}$  for  $\mathbf{u}' \in \mathbb{R}^n$ , we obtain in a similar manner that ( $1 < 1 + \lambda < \alpha$ )

$$E[|\hat{\theta}_k - \theta_k|^{1+\lambda}] = \frac{\lambda \beta \Gamma(1 - \frac{1+\lambda}{\alpha}, \beta + \frac{1+\lambda}{\alpha})}{\sin(\frac{\lambda\pi}{2}) \Gamma(1 - \lambda)} \bar{\sigma}_k^{1+\lambda}, \quad k = 0, 1.$$

#### APPENDIX A. PROOF OF LEMMA 2.1

*Proof.* We express ch.f.  $\varphi(t)$ , given by (2.1), as the product  $\varphi_1(t) \cdot \varphi_2(t)$ , where

$$\begin{aligned} \varphi_1(t) &:= \exp \left\{ i\delta t + \int_{|x| \leq 1} (e^{itx} - 1 - itx) \nu(dx) \right\}, \quad t \in \mathbb{R}, \\ \varphi_2(t) &:= \exp \left\{ \int_{|x| > 1} (e^{itx} - 1) \nu(dx) \right\}, \quad t \in \mathbb{R}. \end{aligned}$$

Since the distribution with ch.f.  $\varphi_1(t)$  has moments of any positive order, it suffices to consider  $\varphi_2(t)$ . We use the necessary and sufficient condition (1.6) for the existence of  $m_\gamma$ , see Lemma 1.4. The following inequalities

$$\begin{aligned} \frac{a}{1+a} &\leq 1 - e^{-a} \leq a, \quad a \geq 0, \\ 1 - \cos b &\leq \frac{b^2}{2} \leq \frac{b}{2}, \quad 0 \leq b \leq 1, \end{aligned}$$

and the fact

$$\int_{|x| > 1} (1 - \cos tx) \nu(dx) \leq \int_{|x| > 1} \left( \frac{(tx)^2}{2} \wedge 1 \right) \nu(dx) := c_t < \infty$$

are used to obtain

$$\begin{aligned} 1 - \operatorname{Re} \varphi_2(t) &\geq 1 - \exp \left\{ \int_{|x| > 1} (\cos tx - 1) \nu(dx) \right\} \geq \frac{1}{1 + c_t} \int_{|x| > 1} (1 - \cos tx) \nu(dx), \\ 1 - \operatorname{Re} \varphi_2(t) &= 1 - \exp \left\{ \int_{|x| > 1} (\cos tx - 1) \nu(dx) \right\} + \exp \left\{ \int_{|x| > 1} (\cos tx - 1) \nu(dx) \right\} \\ &\quad - \cos \left( \int_{|x| > 1} \sin tx \nu(dx) \right) \exp \left\{ \int_{|x| > 1} (\cos tx - 1) \nu(dx) \right\} \end{aligned}$$

$$\leq \int_{|x|>1} (1 - \cos tx) \nu(dx) + \frac{1}{2} \int_{|x|>1} |\sin tx| \nu(dx).$$

Then we notice by Fubini's theorem that

$$\begin{aligned} \int_0^\infty t^{-(1+\gamma)} \int_{|x|>1} (1 - \cos tx) \nu(dx) dt &= \int_0^\infty \frac{1 - \cos v}{v^{1+\gamma}} dv \int_{|x|>1} |x|^\gamma \nu(dx), \\ \int_0^\infty t^{-(1+\gamma)} \int_{|x|>1} |\sin tx| \nu(dx) dt &= \int_0^\infty \frac{|\sin v|}{v^{1+\gamma}} dv \int_{|x|>1} |x|^\gamma \nu(dx). \end{aligned}$$

Since double integrals on the left-hand sides exist if and only if the integrals on the right-hand sides exist, the condition (1.6) is equivalent to the existence of  $\int_{|x|>1} |x|^\gamma \nu(dx)$ .  $\square$

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